The Virtual Casing Principle and Helmholtz's Theorem

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The virtual casing principle[1,2] is used in plasma physics to convert a Biot-Savart integration over a current distribution into a surface integral over a surface that encloses the current. In many circumstances, use of virtual casing to convert a volume integral into a surface integral can significantly speed up the computation of magnetic fields. The virtual casing principle is used for toroidal plasma equilibrium computations, design of tokamak control coils, computation of plasma inductances, magnetic field line tracing, and magnetic diagnostic response calculation.

Previous discussion of the virtual casing principle has been specialized to magnetic (divergence-free) fields, and the argumentation has often relied on properties of a virtual superconductor surrounding the volume in question. Here the virtual casing principle is derived for a general vector field with arbitrary divergence and curl. There is no restriction to curl-free or divergence-free fields.

Consider a general vector field $\boldsymbol{b}$ with curl $\boldsymbol{c}$ and divergence d:

$$\nabla\cdot\mathbf{b}=d \qquad \nabla\times\mathbf{b}=\mathbf{c}$$

Define the related field $\boldsymbol{b}_{\mathrm{V}}$ arising from combined Coulomb and Biot-Savart integration of the field sources over a volume V

$$\mathbf{b}_V(\mathbf{r})\equiv\frac{1}{4\pi}\int_V d(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}d^3\mathbf{r}'+\frac{1}{4\pi}\int_V\frac{\mathbf{c}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|^3}d^3\mathbf{r}'$$

Starting from the divergence theorem applied to a symmetric dyadic,

$$\int_V\nabla\cdot(\mathbf{a}\,\mathbf{b}+\mathbf{b}\,\mathbf{a})d^3\mathbf{r}'=\oint_{\partial V}(\hat{n}'\cdot\mathbf{a})\mathbf{b}+(\hat{n}'\cdot\mathbf{b})\mathbf{a}\;d^2\mathbf{r}'$$

and using standard vector identities with

$$\mathbf{a}\equiv\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3} \qquad \nabla'\times\mathbf{a}=0 \qquad \nabla'\cdot\mathbf{a}=-4\pi\,\delta(\mathbf{r}-\mathbf{r}')$$

one can obtain the virtual casing principle:

$$\mathbf{b}_V(\mathbf{r})=\frac{1}{4\pi}\int_{\partial V}\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}(\hat{n}'\cdot\mathbf{b})-\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}\times(\hat{n}'\times\mathbf{b})\;d^2\mathbf{r}' \quad +\begin{cases}\mathbf{b}(\mathbf{r}) & \mathbf{r}\in V\\ 0 & \mathbf{r}\notin V\end{cases}$$

This general form of the virtual casing principle is closely related to Helmholtz's theorem. Another short calculation shows that the virtual casing principle with a curl-free field is consistent with Green's theorem applied to an electrostatic field.

[1] Shafranov V D and Zakharov L E *Nucl. Fusion* **12** 599-601 (1972)
[2] Lazerson S A *Plasma Phys. Control. Fusion* **54** 122002 (2012)

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